Lecture 8

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JACOBI GEOMETRY II

(references: Chapter 1 of Tortorella's thesis, Vitagliano's work and sections 2.1, 2.3, 2.6 and 2.7 of Zapata's thesis)

Definition Proposition Exercise

(I) LINE BUNDLES AS UNIT-FREE MANIFOLDS

In lecture 7 we introduced the category of line bundles as a natural generalisation of the category of manifold that retained all the essential features but could be interpreted as "unit-free" geometry:

LineMan, submanifolds, quotients, products

$$\begin{tikzcd} L \arrow[r, "b"] \arrow[d] & L' \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd} \quad , \quad \begin{tikzcd} L_S \arrow[r, "\iota"] \arrow[d] & L \arrow[d] \\ S \arrow[r, hook, "i"'] & M \end{tikzcd} \quad , \quad \begin{tikzcd} L \arrow[r, "\pi"] \arrow[d] & L/G \arrow[d] \\ M \arrow[r, "p"'] & M/G \end{tikzcd} \quad , \quad \begin{tikzcd} L_1 \arrow[d] & L_1 \boxtimes L_2 \arrow[l] \arrow[r] \arrow[d] & L_2 \arrow[d] \\ M_1 & M_1 \times M_2 \arrow[l] \arrow[r] & M_2 \end{tikzcd}$$

Conventional Geometry / Unit-less	Unit-Free Geometry
Man	LineMan
rings from C^∞ functor	$C^\infty(M)$-modules from Γ functor
smooth maps $\varphi: M \to N$	factors $\begin{tikzcd} L \arrow[r, "b"] \arrow[d] & L' \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$
pullbacks $\varphi^*: C^\infty(N) \to C^\infty(M)$	pull backs $b^*: \Gamma(L') \to \Gamma(L)$
ring morphism $\varphi^*(fg) = \varphi^*f \cdot \varphi^*g$	module morphisms $b^*(f \cdot s) = \varphi^*f \cdot b^*s$

This encourages us to think of $\mathrm{LineMan}$ as manifolds without the choice of global unit function and to interpret $\Gamma(L)$ as unit-free functions that could represent physical observables for which no unit has been fixed.

We define a local choice of unit or simply a unit as a local non-vanishing section $u \in \Gamma(L^\times)$ defined on some open $\mathcal{U} \subset M$. Indeed, a unit is equivalent to a local trivialisation that will give us an isomorphism of vector bundles (or factor): $L|_{\mathcal{U}} \xrightarrow{\sim} \mathbb{R}_{\mathcal{U}}$.
In particular, we obtain the correspondence:

$$\Gamma(L|_{\mathcal{U}}) = C^\infty(M) \cdot u$$

where for each $s \in \Gamma(L|_{\mathcal{U}})$ there is a unique $f_s \in C^\infty(\mathcal{U})$ such that

$$s = f_s \cdot u .$$

This recovers conventional "unit-less" functions as the coefficients of unit-free functions, we can read the above formula as the unit-free classical observable s being f_s in units of u.

Dynamical notions should now apply to $\Gamma(L)$ instead of $C^\infty(M)$. Following the usual exposition of the theory of manifolds we define local directional derivatives:

$$DL := \{ a_x : \Gamma(L) \to L_x \mid \underset{\mathbb{R}\text{-linear } \&}{a_x(f \cdot s) = f(x) \cdot a_x(s) + \delta(a_x)(f)(x) \cdot s(x)} \}$$

this defines the der bundle of the line bundle L with a map covering the identity $\delta : DL \to TM$. Sections of this bundle induce natural maps on sections:

$$a \in \Gamma(DL) \quad : \quad a : \Gamma(L) \to \Gamma(L) \qquad a[s](x) = a(x)(s)$$
$$s \mapsto a[s]$$

which act as $C^\infty(M)$-module derivations by construction.
We can define a Lie bracket : $[a,b] = a[b[s]] - b[a[s]]$.

Proposition 8.1 $\Gamma(DL) \cong \mathrm{Der}(L)$ as Lie algebras.

proof. Show that δ corresponds to the symbol map on derivations and use the conventional isomorphism of Lie algebra $\Gamma(TM) \cong \mathrm{Der}(C^\infty(M))$.

Given a factor
$$\begin{tikzcd} L \arrow[r, "b"] \arrow[d] & L' \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$
we can define its der map

$$D_x b : D_x L \to D_{\varphi(x)} L'$$
$$a_x \mapsto D_x b(a_x)$$
giving
$$\begin{tikzcd} DL \arrow[r, "Db"] \arrow[d] & DL' \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

via
$$D_x b(a_x)(s') := b_x(a_x(b^* s')) \quad \forall s' \in \Gamma(L').$$

Proposition 8.2 (The Der Functor)

The der bundle and der map construction induce a functor
$$D : \mathrm{Line}_{\mathrm{Man}} \to \mathrm{LVect}_{\mathrm{Man}}$$

i.e. $D(b \circ c) = Db \circ Dc$, $D(\mathrm{id}_L) = \mathrm{id}_{DL}$,

furthermore $Db : DL \to DL'$ is a morphism of Lie algebroids.

proof. (Proposition 2.6.4 in my thesis).

At the level of sections we encounter the notion of b-relatedness and pushforward $b_* : \Gamma(DL) \to \Gamma(DL')$

when b is a factor covering a diffeomorphism.

The der functor in Line is entirely analogous to the tangent functor in Man. See Propositions 2.6.5–2.6.10 where we find:

submanifolds $\qquad$ quotients $\qquad$ products

$$D\iota : DL_S \hookrightarrow DL \qquad D_{[x]}L/G \cong D_xL/\mathfrak{g} \qquad D(L_1 \times L_2) \cong DL_1 \boxplus DL_2$$

$$\delta(DL_S) = TS$$

The der bundle carries a natural short exact sequence given by the anchor δ:

$$0 \longrightarrow \mathbb{R}_M \longrightarrow DL \xrightarrow{\delta} TM \longrightarrow 0$$

where we have observed that $\operatorname{Ker}\delta = \operatorname{End}(L) \cong \mathbb{R}_M$.

In analogy with the cotangent bundle in ordinary geometry we can consider the L-dual of the der bundle in the category of LV bundles to obtain the following sequence:

$$0 \longleftarrow L \longleftarrow (DL)^{*L} \xleftarrow{\delta^{*id_L}} (TM)^{*L} \longleftarrow 0$$

This is indeed the bundle version of the Spencer sequence for jet bundles:

$$0 \leftarrow L \xleftarrow{\pi^1} J^1L \xleftarrow{i} T^*M \otimes L \leftarrow 0$$

Thus we encounter the jet bundle of a line bundle naturally as the L-dual of its der bundle. The jet bundle J^1L is such that we can define a "unit-free" differential

$$j^1 : \Gamma(L) \longrightarrow \Gamma(J^1L)$$

such that we can write: $\qquad$ _ action by L-duality.

$a \in \Gamma(DL)$: $\quad a[s] = j^1 s(a)$

this is called the jet map or jet prolongation.

At this point we can identify the multiderivations and L-forms just like we did in the linear case in lecture 7:

$$\mathfrak{X}^\bullet(L) := \Gamma(\Lambda^\bullet(DL)) \quad \text{is a } (\mathfrak{X}^\bullet(M), \wedge)\text{-module}$$

$$\Omega_L^\bullet(L) := \Gamma(\Lambda^\bullet(DL)^* \otimes L) \quad \text{is a } (\Omega^\bullet(M), \wedge)\text{-module}$$

The bracket on $\Gamma(DL) = \mathfrak{X}^1(L)$ extends to a Gerstenhaber-like bracket on $\mathfrak{X}^\bullet(L)$ and the jet $j^1 : \Omega_L^0(L) \to \Omega_L^1(L)$ extends to a graded module differential which allows to define the Cartan calculus on L (which turns out to be just the Lie algebroid Cartan calculus on DL with values in L).

Upon a choice of units, all the notions above recover the ordinary notions in smooth manifolds:

in the case of $L = \mathbb{R}_M$ we have:

$$DL \cong TM \oplus \mathbb{R}_M$$

$$J^1 L \cong T^*M \oplus \mathbb{R}_M$$

$$[X \oplus f, Y \oplus g] = [X, Y] \oplus X[g] - Y[f]$$

$$j^1 = d \oplus \mathrm{id}_{C^\infty(M)} .$$

II JACOBI MANIFOLDS

Motivated by Proposition 6.4 (rk 1 local Lie algebras) and the discussion on line bundles so far, we define Jacobi structures as the unit-free analogue of Poisson manifolds:

a Jacobi structure is a local Lie algebra structure on the section of a line bundle $(\Gamma(L), \{,\})$.

Since $\mathrm{Diff}_1(L) = \mathrm{Der}(L)$, a Jacobi bracket on sections is equivalently given by a Jacobi biderivation, which is a section of biderivations $\Pi \in \Gamma(\Lambda^2(DL))$ satisfying the Gerstenhaber algebra integrability condition $[\![\Pi, \Pi]\!] = 0$. The bracket is recovered as:

$$\{s, r\} = \Pi(j^1 s, j^1 r).$$

Regarding these objects in the appropriate categories of line bundles and LV bundles, we see that a Jacobi structure carries a musical map that now manifests as the following commutative diagram:

$$\begin{array}{ccc} J^1L & \xrightarrow{\Pi^\#} & DL \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle \delta} \\ T^{*L}M := T^*M\otimes L & \xrightarrow[\Lambda^\#]{} & TM \end{array}$$

At the level of sections, the above constructions align with the symbol-squiggle construction for general derivative Lie algebras to give the Hamiltonian maps:

$$D : \Gamma(L) \to \Gamma(DL)$$
$$\Pi^\# : \Gamma(J^1L) \to \Gamma(DL)$$
$$X : \Gamma(L) \to \Gamma(TM)$$

$$\Lambda^\#: \Gamma(T^{*L}M) \to \Gamma(TM)$$

$D_s := \{s,-\} = \pi^\#(j's)$, $D_{f\cdot s} = f\cdot D_s + \pi^\#(i(df\otimes s))$

$X_s := \sigma_{D_s}$, $D_s(f\cdot r) = f\cdot D_s(r) + X_s[f]\cdot r$

$\Lambda^\# := \sigma_X$, $X_{f\cdot s} = f\cdot X_s + \Lambda^\#(df\otimes s)$

Let two Jacobi structures $(L,\{,\})$ and $(L',\{,\}')$, a factor $b: L\to L'$ is called a Jacobi map if

$\forall\, s,r\in\Gamma(L')$ $\qquad b^*\{s,r\}' = \{b^*s, b^*r\}$.

III RECOVERING KNOWN STRUCTURES VIA CHOICE OF UNIT

Let $\left(\begin{smallmatrix}L\\ \downarrow\\ M\end{smallmatrix}, \{,\}\right)$ be a Jacobi structure and let $u\in\Gamma(L^\times)$ be a choice of unit defined on an open $\mathcal{U}\subset M$.

The choice of $u\in\Gamma(L^\times)$ induces a vector fied and bivector on local functions:

$$R[f] := X_u[f] \ , \quad \pi(df,dg) := \Lambda(df\otimes u, dg\otimes u)$$

Proposition 8.3 (Trivialised Jacobi structures)

The pair (R,π) endows $\mathcal{U}$ with a Lichnerowicz structure

$$[\![R,\pi]\!]=0 \quad \& \quad [\![\pi,\pi]\!] + 2R\wedge\pi = 0$$

proof. Use the fact that all local sections are of the form $C^\infty(\mathcal{U})\cdot u$ and use the symbol-squiggle theorem. The two conditions of a Lichnerowicz structure follow directly from conditions iii) and iv) in the symbol-squiggle theorem.

It follows from this result that the unit-preserving functions

$$C^\infty_u(\mathcal{U}) := \{ f \in C^\infty(M) : X_u[f] = 0 \}$$

always inherit a Poisson bracket $(C^\infty_u(\mathcal{U}), \{,\}_u)$ which is explicitly defined from the identity:

$f, g \in C^\infty_u(\mathcal{U})$; $\{f,g\}_u \cdot u = \{ f \cdot u, g \cdot u \}$.

In a complementary direction, we define $u \in \Gamma(L^\times)$ to be a Poisson unit if $X_u = 0$, since in this case the Lichnerowicz structure induced in $C^\infty(\mathcal{U})$ has $R = 0$ and is thus Poisson.

General Jacobi manifolds do not admit Poisson units (see V).
The identification of Poisson units allows for a proper characterisation of the category of Poisson manifolds as a subcategory of Jacobi manifolds via the following definitions:

conformal Poisson structure: Jacobi structure that admits covering trivialisations via Poisson units.

Poisson structure: Jacobi structure that admits a global Poisson unit.

IV CONSTRUCTIONS IN JACOBI MANIFOLDS

(see Section 2.7 of my thesis)

Coisotropic submanifold $S \hookrightarrow M$ (with $L \to M$)

$$S^{0L} := (DL_S)^{0L} \subset (J^1 L, \Pi) \text{ is isotropic } \Pi|_{S^{0L}} = 0$$

this is equivalent to

vanishing sections $\Gamma_S \subset \Gamma(L)$ being a Lie sub algebra $\{\Gamma_S, \Gamma_S\} \subset \Gamma_S$

product Jacobi structure

$(L_{1,2}, \{,\}_{1,2})$ $\exists! \ (L_1 \boxtimes L_2, \{,\}_{12})$ such that

$$\{P_i^* s, P_i^* r\}_{12} = P_i^* \{s,r\}_i \quad \text{and} \quad \{P_1^* s, P_2^* s'\}_{12} = 0$$

Coisotropic relations a factor $b : L_1 \to L_2$ defines the Lineman analogue of a graph $L\mathrm{graph}(b) \subset M_1 \boxtimes M_2$. then

b is Jacobi map $\Leftrightarrow$ $L\mathrm{graph}(b)$ is coisotropic in $L_1 \boxtimes \bar{L}_2$.

Proposition 8.4 (Coisotropic reduction)

$(L, \{,\})$ Jacobi and C coisotropic with reduction diagram:

$$\begin{tikzcd}
L_C \arrow[r, "\iota"] \arrow[d, "\pi"'] \arrow[dr] & L \arrow[dr, "s"] & \\
L' \arrow[dr] & C \arrow[r, hook, "i"] \arrow[d, "p"] & M \\
& M' &
\end{tikzcd}$$

such that

$$\delta(\mathrm{Ker}\, D\pi) = \Lambda^\#(TC^{0L})$$

then $\exists! \ (L', \{,\}')$ such that $(L, \{,\}) \overset{\text{reduces}}{\dashrightarrow} (L', \{,\}')$.

proof. (Proposition 2.7.6 in my thesis)

Jacobi submanifolds as coisotropic reductions with trivial projection etc.

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(V) PRECONTACT AND NON-DEGENERATE JACOBI STRUCTURES

(see end of section 2.7 of my thesis)

Precontact structure pair $(M, H \subset TM)$ H hyperplane.

projection form $\theta : TM \to TM/H =: L$, $H = \ker\theta$

curvature form $\eta : \Gamma(\wedge^2 H) \to \Gamma(L)$, $\eta^\flat : H \to H^{*L}$

$(X, Y) \mapsto \theta([X,Y])$

contact structure

H integrable $\Rightarrow$ $\eta^\flat = 0$

H maximally non-integrable $\Rightarrow$ $\eta^\flat : H \xrightarrow{\sim} H^{*L}$ isomorphism

isotropic submanifold $S \subset M$, $TS \subset H$.

Proposition 8.5

$$\left\{ \begin{matrix} \text{contact} \\ \text{structures} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{non-degenerate} \\ \text{Jacobi structures} \end{matrix} \right\}$$

proof. (Proposition 2.7.8 in my thesis)

one direction is clear by setting $H := \Lambda^\#(T^{*L}M) \subset TM$

the other direction involves a bit more work setting

$$\Lambda^\#(df \otimes u) := \eta^\#(df|_H \otimes u).$$

The category of contact manifolds

identify **Legendrian submanifolds** as those isotropic and coisotropic, morphisms are Legendrian relations defined on product contact manifolds.