

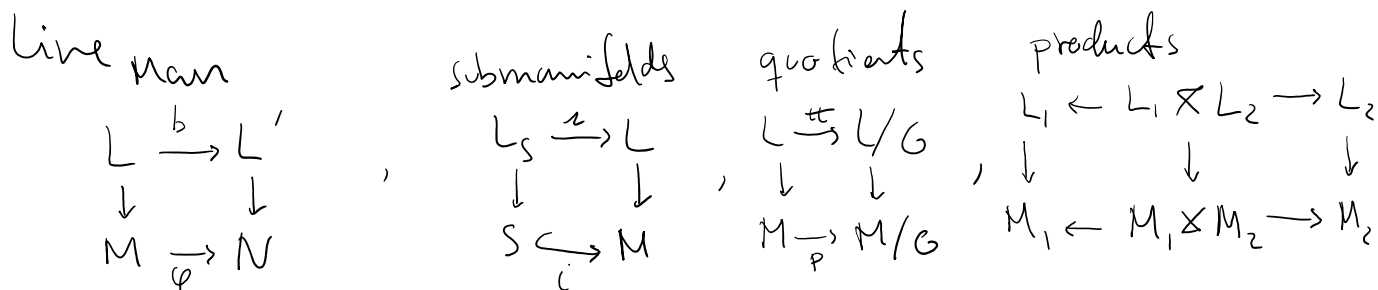
JACOBI GEOMETRY II

(references: Chapter 1 of Tortorella's thesis, Vitagliano's work and sections 2.1, 2.3, 2.6 and 2.7 of Zapata's thesis)

Definition Proposition Exercise

(I) LINE BUNDLES AS UNIT-FREE MANIFOLDS

In lecture 7 we introduced the category of line bundles as a natural generalisation of the category of manifold that retained all the essential features but could be interpreted as "unit-free" geometry:



Conventional Geometry / Unit-less

Man

rings from C^∞ functor

smooth maps $\varphi: M \rightarrow N$

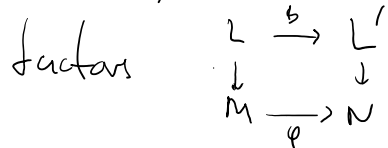
pullbacks $\varphi^*: C^\infty(N) \rightarrow C^\infty(M)$

ring morphism $\varphi^*(fg) = \varphi^*f \cdot \varphi^*g$

Unit-Free Geometry

Line man

$C^\infty(M)$ -modules from Γ functor



pull backs $b^*: \Gamma(L') \rightarrow \Gamma(L)$

module morphism $b^*(f \cdot s) = \varphi^*f \cdot b^*s$

This encourages us to think of Line_M as manifolds without the choice of global unit function and to interpret $\Gamma(L)$ as unit-free functions that could represent physical observables for which no unit has been fixed.

We define a **local choice of unit** or simply a **unit** as a local non-vanishing section $u \in \Gamma(L^\times)$ defined on some open $U \subset M$.

Indeed, a unit is equivalent to a local trivialisation that will give us an isomorphism of vector bundles (or factors): $L|_U \xrightarrow{\sim} \mathbb{R}u$.

In particular, we obtain the correspondence:

$$\Gamma(L|_U) = C^\infty(U) \cdot u$$

where for each $s \in \Gamma(L|_U)$ there is a unique $f_s \in C^\infty(U)$ such that

$$s = f_s \cdot u$$

This recovers conventional "unit-less" functions as the coefficients of unit-free functions, we can read the above formula as the unit-free classical observable s being f_s in units of u .

Dynamical notions should now apply to $\Gamma(L)$ instead of $C^\infty(M)$. Following the usual exposition of the theory of manifolds

we define **local directional derivatives**:

$$DL := \left\{ a_x: \Gamma(L) \rightarrow L_x \mid a_x(f \cdot s) = f(x) \cdot a_x(s) + \overset{\mathbb{R}\text{-linear}}{D(a_x)(f)}(x) \cdot s(x) \right\}$$

this defines the **der bundle** of the line bundle L with a map covering the identity $\mathcal{J}: DL \rightarrow TM$. Sections of this bundle induce natural maps on sections:

$$a \in \Gamma(DL) : a : \Gamma(L) \rightarrow \Gamma(L) \quad a[s](x) = a(x)(s)$$

$$s \mapsto a[s]$$

which act as $C^\infty(M)$ -module derivations by construction.
 We can define a Lie bracket: $[a, b] = a[b[s]] - b[a[s]]$.

Proposition 8.1 $\Gamma(DL) \cong \text{Der}(L)$ as Lie algebras.

proof. Show that \mathcal{L} corresponds to the symbol map on derivations and use the conventional isomorphism of Lie algebra $\Gamma(TM) \cong \text{Der}(C^\infty(M))$.

Given a factor $\begin{array}{ccc} L & \xrightarrow{b} & L' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array}$ we can define its *der map*

$$D_x b : D_x L \rightarrow D_{\varphi(x)} L' \quad \text{giving} \quad \begin{array}{ccc} DL & \xrightarrow{D_b} & DL' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array}$$

$$a_x \mapsto D_x b(a_x)$$

via

$$D_x b(a_x)(s') := b_x(a_x(b^* s')) \quad \forall s' \in \Gamma(L')$$

Proposition 8.2 (The Der Functor)

The der bundle and der map construction induce a functor

$$D : \text{Lin}_{\text{man}} \rightarrow \text{LVect}_{\text{man}}$$

$$\text{i.e. } D(b \circ c) = D_b \circ D_c, \quad D(\text{id}_L) = \text{id}_{DL},$$

furthermore $D_b : DL \rightarrow DL'$ is a morphism of Lie algebroids.

proof. (Proposition 2.6.4 in my thesis).

At the level of sections we encounter the notion of *b-relatedness* and *pushforward* $b_* : \Gamma(DL) \rightarrow \Gamma(DL')$

when b is a factor covering a diffeomorphism.

The der functor in line is entirely analogous to the tangent functor in Man. See Propositions 2.6.5-2.6.10 where we

find:

submanifolds

quotients

products

$$D_1: DL_S \hookrightarrow DL$$

$$D_{[x]} L/G \cong D_x L/g$$

$$DL_1 \times DL_2 \cong DL_1 \oplus DL_2$$

$$\delta(DL_S) = TS$$

The der bundle carries a natural short exact sequence given by the anchor δ :

$$0 \rightarrow \mathbb{R}M \rightarrow DL \xrightarrow{\delta} TM \rightarrow 0$$

where we have observed that $\ker \delta = \text{End}(L) \cong \mathbb{R}M$.

In analogy with the cotangent bundle in ordinary geometry we can consider the L -dual of the der bundle in the category of LV bundles to obtain the following sequence:

$$0 \leftarrow L \leftarrow (DL)^{*L} \xleftarrow{\delta^{*id_L}} (TM)^{*L} \leftarrow 0$$

This is indeed the bundle version of the Spencer sequence for jet bundles:

$$0 \leftarrow L \xleftarrow{\pi^1} J^1 L \xleftarrow{i} T^*M \otimes L \leftarrow 0$$

Thus we encounter the **jet bundle** of a line bundle naturally as the L -dual of its der bundle. The jet bundle $J^1 L$ is such that we can define a "unit-free" differential

$$j^1: \Gamma(L) \rightarrow \Gamma(J^1 L)$$

such that we can write:

— action by L -duality.

$$a \in \Gamma(DL) : a[S] = j^1 s(a)^\leftarrow$$

this is called the **jet map** or **jet prolongation**.

At this point we can identify the **multiderivations** and **L-forms** just like we did in the linear case in lecture 7:

$$\mathcal{X}^*(L) := \Gamma(\wedge^*(DL)) \quad \text{is a } (\mathcal{X}^*(M), \wedge) \text{-module}$$

$$\Omega_i^*(L) := \Gamma(\wedge^*(DL)^* \otimes L) \quad \text{is a } (\Omega_i^*(M), \wedge) \text{-module}$$

The bracket on $\Gamma(DL) = \mathcal{X}^*(L)$ extends to a Gerstenhaber-like bracket on $\mathcal{X}^*(L)$ and the jet $j^1: \Omega_1^0(L) \rightarrow \Omega_1^1(L)$ extends to a graded module differential which allows to define the **Cartan calculus** on L (which turns out to be just the Lie algebroid Cartan calculus on DL with values in L).

Upon a choice of units, all the notions above recover the ordinary notions in smooth manifolds:
in the case of $L = \mathbb{R}_M$ we have:

$$DL \cong TM \oplus \mathbb{R}_M$$

$$j^1 L \cong T^*M \oplus \mathbb{R}_M$$

$$[X \oplus f, Y \oplus g] = [X, Y] \oplus X[g] - Y[f]$$

$$j^1 = d \oplus \text{id}_{\mathbb{R}_M}.$$

II JACOBI MANIFOLDS

Motivated by Proposition 6.4 (rk 1 local Lie algebras) and the discussion on line bundles so far, we define **Jacobi structures** as the unit-free analogue of Poisson manifolds:

a Jacobi structure is a local Lie algebra structure on the sections of a line bundle $(\Gamma(L), \{, \})$.

Since $\text{Diff}_1(L) = \text{Der}(L)$, a Jacobi bracket on sections is equivalently given by a **Jacobi biderivation**, which is a section of biderivations $\Pi \in \Gamma(\Lambda^2(DL))$ satisfying the Gerstenhaber algebra integrability condition $[\Pi, \Pi] = 0$. The bracket is recovered as:

$$\{s, r\} = \Pi(j^1s, j^1r).$$

Regarding these objects in the appropriate categories of line bundles and LV bundles, we see that a Jacobi structure carries a **musical map** that now manifests as the following commutative diagram:

$$\begin{array}{ccc} J^1L & \xrightarrow{\Pi^\#} & DL \\ i \uparrow & & \downarrow \delta \\ T^*M := T^*M \otimes L & \xrightarrow{\wedge^\#} & TM \end{array}$$

At the level of sections, the above constructions align with the symbol-squiggle construction for general derivative Lie algebras to give the **Hamiltonian maps**:

$$\begin{aligned} D &: \Gamma(L) \rightarrow \Gamma(DL) \\ \Pi^\# &: \Gamma(J^1L) \rightarrow \Gamma(DL) \\ X &: \Gamma(L) \rightarrow \Gamma(TM) \end{aligned}$$

$$\Lambda^\# : \Gamma(T^*L^*M) \rightarrow \Gamma(TM)$$

$$D_s := \{s, -\} = \pi^\#(j^!s), \quad D_{f \cdot s} = f \cdot D_s + \pi^\#(i(df \otimes s))$$

$$X_s := \sigma_{D_s}, \quad D_s(f \cdot r) = f \cdot D_s(r) + X_s[f] \cdot r$$

$$\Lambda^\# := \sigma_X, \quad X_{f \cdot s} = f \cdot X_s + \Lambda^\#(df \otimes s)$$

Let two Jacobi structures $(L, \{, \})$ and $(L', \{, \}')$, a factor $b: L \rightarrow L'$ is called a **Jacobi map** if

$$\forall s, r \in \Gamma(L') \quad b^* \{s, r\}' = \{b^*s, b^*r\}.$$

III RECOVERING KNOWN STRUCTURES VIA CHOICE OF UNIT

Let $(\begin{smallmatrix} L \\ L' \end{smallmatrix}, \{, \})$ be a Jacobi structure and let $u \in \Gamma(L^*)$ be a choice of unit defined on an open $U \subset M$.

The choice of $u \in \Gamma(L^*)$ induces a vector field and bivector on local functions:

$$R[f] := Xu[f], \quad \pi(df, dg) := \Lambda(df \otimes u, dg \otimes u)$$

Proposition 8.3 (Trivialised Jacobi structures)

The pair (R, π) endows U with a Lichnerowicz structure

$$[R, \pi] = 0 \quad \& \quad [\pi, \pi] + 2R \wedge \pi = 0$$

proof. Use the fact that all local sections are of the form $C^\infty(U) \cdot u$ and use the symbol-squiggle theorem. The two conditions of

a Lichnerowicz structure follow directly from conditions iii) and iv) in the symbol-squiggle theorem.

It follows from this result that the unit-preserving functions

$$C_u^\infty(\mathcal{U}) := \{f \in C^\infty(\mathcal{U}) : X_u[f] = 0\}$$

always inherit a Poisson bracket $(C_u^\infty(\mathcal{U}), \{, \}_u)$ which is explicitly defined from the identity:

$$f, g \in C_u^\infty(\mathcal{U}) \quad ; \quad \{f, g\}_u \cdot u = \{f \cdot u, g \cdot u\}.$$

In a complementary direction, we define $u \in \Gamma(L^X)$ to be a **Poisson unit** if $X_u = 0$, since in this case the Lichnerowicz structure induced in $C^\infty(\mathcal{U})$ has $R=0$ and is thus Poisson.

General Jacobi manifolds do not admit Poisson units (see \textcircled{V}). The identification of Poisson units allows for a proper characterization of the category of Poisson manifolds as a subcategory of Jacobi manifolds via the following definitions:

conformal Poisson structure: Jacobi structure that admits covering trivialisations via Poisson units.

Poisson structure: Jacobi structure that admits a global Poisson unit.

\textcircled{IV} CONSTRUCTIONS IN JACOBI MANIFOLDS

(see Section 2.7 of my thesis)

Coisotropic submanifold $S \hookrightarrow M$

$$S^{0L} := (DL_S)^{0L} \subset (J^1L, \Pi) \text{ is isotropic } \Pi|_{S^{0L}} = 0$$

this is equivalent to

vanishing sections $\Gamma_S \subset T(L)$ being a Lie subalgebra $\{\Gamma_S, \Gamma_S\} \subset \Gamma_S$

product Jacobi structure

$(L_{1,2}, \{\cdot, \cdot\}_{1,2}) \exists! (L_1 \times L_2, \{\cdot, \cdot\}_{1,2})$ such that

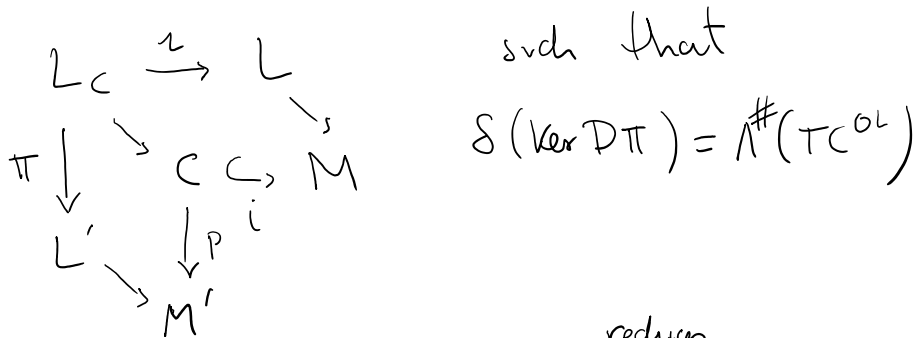
$$\{P_i^* s, P_i^* r\}_{1,2} = P_i^* \{s, r\}_i \quad \text{and} \quad \{P_1^* s, P_2^* s'\}_{1,2} = 0$$

Coisotropic relations a factor $b: L_1 \rightarrow L_2$ defines the Lie algebra analogue of a graph $L_{\text{graph}(b)} \subset M_1 \times M_2$, then

b is Jacobi map $\iff L_{\text{graph}(b)}$ is coisotropic in $L_1 \times \bar{L}_2$.

Proposition 8.4 (Coisotropic reduction)

$(L, \{\cdot, \cdot\})$ Jacobi and C coisotropic with reduction diagram:



$$S(\text{Ker } D\pi) = \Lambda^\#(TC^{0L})$$

then $\exists! (L', \{\cdot, \cdot\}')$ such that $(L, \{\cdot, \cdot\}) \xrightarrow{\text{reduces}} (L', \{\cdot, \cdot\}')$.

proof. (Proposition 2.7.6 in my thesis)

Jacobi submanifolds as coisotropic reduction with trivial projection id .

Ⓟ PRECONTACT AND NON-DEGENERATE JACOBI STRUCTURES

(see end of section 2.7 of my thesis)

precontact structure pair $(M, H \subset TM)$ H hyperplane.

projection form $\theta: TM \rightarrow TM/H =: L$, $H = \ker \theta$

curvature form $\eta: \Gamma(\wedge^2 H) \rightarrow \Gamma(L)$ $\eta^b: H \rightarrow H^*L$
 $(X, Y) \mapsto \theta([X, Y])'$

contact structure

H integrable $\Rightarrow \eta^b = 0$

H maximally non-integrable $\Rightarrow \eta^b: H \xrightarrow{\sim} H^*L$ isomorphism

isotropic submanifold $S \subset M$, $TS \subset H$.

Proposition 8.5

$\left\{ \begin{array}{l} \text{contact} \\ \text{structures} \end{array} \right\} \text{ : : } \left\{ \begin{array}{l} \text{non-degenerate} \\ \text{Jacobi structures} \end{array} \right\}$

proof. (Proposition 2.7.8 in my thesis)

one direction is clear by setting $H := \Lambda^\#(T^*LM) \subset TM$

the other direction involves a bit more work setting

$$\Lambda^\#(df \otimes u) := \eta^\#(df|_H \otimes u).$$

the category of contact manifolds

identify Legendrian submanifolds as those isotropic and coisotropic, morphisms are Legendrian relations defined on product contact manifolds.